Dynamic and Debye shielding and anti-shielding

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While shielding in collisional plasmas obeys the standard Debye result, shielding in collisionless plasmas is far more complex than commonly believed. For example, a one-dimensional (highly magnetized), immobile-ion plasma can, in some circumstances, anti-shield a positive test charge; i.e. the plasma becomes more positive in the vicinity of the test charge. When shielding does occur, it results from electron dynamics. Collisionless shielding in one, two and three dimensions is presented here, and is in excellent agreement with experiments in pure electron plasmas. Because the distribution functions found in Dynamic shielding are highly non-Maxwellian in the non-linear regime, collisionless Dynamic shielding can be substantially less efficacious than collisional Debye shielding. © 1996 American Institute of Physics. [S1070-664X(96)94205-8]

I. INTRODUCTION

One of the most fundamental plasma characteristics is that plasmas shield applied electrostatic perturbations. A test charge placed in a plasma will be surrounded by a cloud of oppositely charged plasma particles, which both shields the remaining plasma from the test charge and lowers the electrostatic potential induced by the test charge. The special case of collisional, or Debye shielding, in which the plasma remains Maxwellian throughout, is well understood. Moreover, it is commonly believed that collisions are not actually necessary for shielding; this point of view is so ubiquitous that the requirement that the plasma be Maxwellian is often not made explicit. Collisionless shielding is paradoxical, however. As electrons accelerate towards a positive test charge, flux conservation demands that their density decrease. Consequently, a positive test charge can be surrounded by a net positively charged plasma, and both the fields from the test charge and the potential in the vicinity of the test charge can be accentuated. In this case a plasma can actually anti-shield a test charge.

That one-dimensional (1-D), collisionless plasmas anti-shielding has been recognized in one group of papers on collisionless shielding. We present here what we believe is the first direct experimental evidence of this phenomenon. According to this group of papers, two-dimensional (2-D), collisionless plasmas should neither shield nor anti-shield; the field from a test line charge is predicted to be unmodified by a collisionless plasma. Three-dimensional (3-D) plasmas are not predicted by this group to shield test charges, albeit somewhat weakly. Other sources predict that plasmas of all dimensions will shield, but their results are inadequately justified. All these papers ignore the existence of non-collisional trapping mechanisms: two mechanisms of particular importance are transit-time or adiabatic trapping and instantaneous trapping. We will show that these mechanisms cannot be ignored, and that collisionless plasmas will normally shield a test charge, in accordance with common belief and the second group of papers. This conclusion is verified by the pure-electron plasma experiments described at the end of this paper. Finally, a third group of papers analyze the instantaneous application of a test potential, and predict that 3-D plasmas will shield. As discussed later, however, the results of these papers are limited to small applied potentials and are valid only in the restricted time range $1/\omega_p < t < 1/\omega_p$, where $\omega_p$ is the plasma frequency and $\omega_p$ is the bounce frequency of electrons trapped in the applied potential.

Since collisionless shielding depends on particle dynamics rather than on particle collisions, we call the process Dynamic shielding. Similar, but more complex phenomena such as double layers, Bernstein, Greene and Kruskal (BGK) modes, virtual cathodes, and sheaths have been studied extensively. Indeed, most of the concepts required to analyze Dynamic shielding have been described before, most notably by Gurevich, who analyzed non-collisional trapping mechanisms and self-consistent potential solutions, but did not consider the problem of shielding. Given all this prior work, it is surprising that adiabatic-collisionless shielding itself has not been properly analyzed before.

II. THEORY

While shielding is often conceived to be the masking of the field of a test charge, a more general description is the partial neutralization of an imposed test potential well. This test well may, for instance, be created by biasing a grid in a neutral plasma or by biasing a confinement electrode in a non-neutral plasma. However the well is created, the plasma responds to the net potential: the sum of the test potential and the plasma potential. Since the plasma potential depends, through changes in the plasma density, on the net potential, the response of the plasma must be found by self-consistently solving for the net potential. We will show that the solution depends on the well time history, and the plasma dimensionality and longitudinal extent.

A. One-dimensional adiabatic theory

Energy conservation requires that the velocity of untrapped electrons, subjected to a positive test potential well,
increase according to the dimensionless relation 
\[ v^2 = v_0^2 + 2\Phi, \]
where the normalized velocity \( v \) is defined such that \( v(T/m)^{1/2} \) is the velocity inside the well, \( v_0 \) is the velocity outside the well similarly normalized, \( m \) is the mass of the electron, \( T \) is the plasma temperature, \( kT/\epsilon \) is the self-consistent well depth, and \( -\epsilon \) is the charge on an electron. The velocity \( v_0 \) is a constant of the motion; since any function of a constant of the motion is a solution of the Vlasov equation, the distribution of electrons in the well is
\[ f(v) = f_0[v_0(v_0)] = f_0[(v^2 - 2\Phi)^{1/2}], \]
where \( f_0(v_0) \) is the initial distribution. If the electrons come from an infinite length, collisionless, Maxwellian plasma source, then \( f_0 \) is a Maxwellian distribution at temperature \( T \) and density \( n_0 \). The density of untrapped electrons as a function of the well depth is found by integrating \( f(v) \) from the minimum electron velocity inside the well, \( (2\Phi)^{1/2} \) to infinity, and is given by
\[
n_\text{el}(\Phi) = n_0 \exp(\Phi) \text{erfc}(\sqrt{\Phi}) \\
= n_0 (1 - 2\sqrt{\Phi/\pi} + \Phi + \ldots),
\]
where \( \text{erfc} \) is the complementary error function. This free density, \( n_\text{el}(\Phi) \) is the density those electrons untrapped by the potential well. Since \( n_\text{el}(\Phi) \) is always less than \( n_0 \), the free electrons can only anti-shield a positive external perturbation.

The conclusion that the plasma will anti-shield\(^7\) ignores the possibility that electrons could be trapped in the well by non-collisional mechanisms, thereby increasing the local plasma density. In particular, the conclusion ignores the question of how the well was created. If the well is created instantaneously, slow electrons that happen to be in the well at the time of its creation will be trapped. If, as is more likely, the well is created adiabatically, the transit-time trapping mechanism described below will trap electrons. In fact, the only way to avoid trapping electrons is for the well to be created before electrons are allowed into the vicinity of the well or to use a non-monotonic initial distribution function. Only in these limited circumstances do experiments show that the plasma anti-shields.

Transit-time trapping in an infinite length plasma was first discussed by Gurevich.\(^1\) The process is most easily visualized for square test wells, but occurs for any shape well. Consider a slow-moving (initial velocity \( v_0 \)) electron. Although it gains kinetic energy on entering a slowly-deepening test well, its kinetic energy remains constant inside the well, despite the increasing test well depth. If the well depth increases sufficiently during the transit time, the electron may not have sufficient energy to climb out of the well—the electron is trapped. Further changes in the well depth do not change the kinetic energy; the electron’s total energy simply varies proportionally as the well depth is increased. Hence the electron’s initial velocity \( v_0 \) is a constant of the motion, and the distribution of electrons in the well is given by \( f_0(V) \). Further, if the well depth increases very slowly, only particles with velocity \( v_0 \) will be trapped, so their distribution function reduces to \( f_0(0) \) (see Fig. 1). Integrating over the trapped electron distribution function gives the trapped density

\[ n_\text{tr}(\Phi) = 2n_0 \sqrt{\Phi/\pi}. \]
Far from the charged plane both Debye and Dynamic shielded potentials decay exponentially with scale length $\lambda_D$. Near the plane the potential in the Dynamic case can be significantly greater than in the Debye case because $n_{De}(\Phi) \approx n_{De}(\Phi)$. 

The collisionless response of the plasma to a negative well is substantially simpler than to a positive well because the negative well—effectively a potential hill—cannot trap electrons. In an infinite length plasma, the electron density decreases as $\Phi < 0$. Since this matches the Boltzmann relation, the resulting shielding will be identical to the standard Debye response.

Since positive ions are not trapped by positive wells, ions with charge $q$ will respond to positive wells as $n_p \exp(-q\Phi/e)$. Consequently, Dynamic-electron shielding will eventually be supplemented by Debye-like ion shielding when the ions are mobile.

Shielding in pure-electron plasmas is complicated by three-dimensional effects. Even though individual electrons are tied to magnetic field lines and behave one dimensionally, the response along different field lines differs, thereby suggesting a 3-D model. We have implemented such a 3-D model numerically, but have found that, for our particular experimental parameters, a simplified 1-D model is in good agreement with the 3-D numerical solutions and is sufficient to explain our data.

Our 1-D model starts by assuming that all electrons, regardless of their radial position, respond to a common self-consistent well potential

$$\Phi = \Phi_0 + \Delta \Phi_0 (\Phi).$$

This assumption implies that the plasma radius is both small compared to the wall radius and on the same order as the Debye length, conditions met by our experiment. Here $\Phi_0 = kT \Phi_0/e$ is the externally-applied well potential, and $kT \Delta \Phi_0/e$ is the potential difference between the inside and the outside of the wall that results from changes in the plasma density.

$$\Delta \Phi_0 (\Phi) = \frac{\phi(\Phi, r) - \phi(0, r)}{r},$$

where $kT \phi(\Phi, r)/e$ is the potential of the plasma of density $n_{De}(\Phi)$, $kT \phi(0, r)/e$ is the potential of a plasma of density $n_0(r)$, and $\langle \rangle$ denotes an average over the radius $r$, weighted by the plasma density. Since $\phi(\Phi, r)$ is a linear function of the density,

$$\Delta \Phi_0 (\Phi) = C \{n_{De}(\Phi)/n_0 - 1\},$$

where the density proportionality constant $C$ is found by computing the weighted average. The complete response of the plasma to an external perturbation is found by self-consistently solving Eq. (8). To lowest order in $\Phi$, the solution of Eq. (8) is

$$\Phi = \Phi_{0}/(1 + |\Phi| + \Phi_{0}/(1 + |\Phi|) e V p / e k T).$$

The density inside the well is $n_{De} - 1/2$.

$$\Phi_{0} = \Phi_{0}/(1 + |\Phi| + \Phi_{0}/(1 + |\Phi|) e V p / e k T).$$

Here we have used $C = e V p / e k T = r_{p} / \lambda_{D}^{2}$, where $V_p$ is the central potential of the plasma cylinder and $r_p$ is the plasma radius. The shielding efficacy, $1 - \Phi/\Phi_{0}$, is plotted in Figs. 3 and 4. Note that perfect shielding corresponds to a shielding efficiency of one, and anti-shielding corresponds to negative shielding efficacies.

The collisionally-relaxed Debye response is found by replacing $n_{De}$ with $n_{De}$. For small applied potentials, the plasma again responds as $n_{De}/n_{De} = 1 + \Phi_0/(1 + e V p / e k T)$. However, since $n_0(\Phi) \approx n_{De}(\Phi)$, the Debye response will once again be stronger than the Dynamic response.

This same formalism applies to the anti-shielding case. Here, the self-consistent potential reduces to

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FIG. 2. Comparison of Debye and Dynamic shielding of a charged plane.

FIG. 3. Shielding efficacy $1 - \Phi/\Phi_{0}$, vs. applied well voltage $V_p$ at $T = 5 \times 10^{-6}$.
\[ n_s(\Phi) = n_0 \{ \exp(\Phi) + \text{erf}(\sqrt{\Phi})[1 - \exp(\Phi)] \} = n_0(1 + \Phi + \ldots). \quad (13) \]

The degree of shielding is found by substituting this density into Eq. (6) or (10).

While Instantaneous shielding yields the same linear response as either Debye or Dynamic shielding, the nonlinear response is far weaker, as is reflected in the low shielding efficiencies graphed in Figs. 3 and 4, where these shielding efficiencies are calculated using approximate density given by Eq. (13). Shielding is normally thought to work best in the low temperature limit, and, indeed, the (infinite length) shielding efficiency for both Debye and Dynamic shielding goes to one in this limit. However, it is easy to show that the Instantaneous shielding efficiency scales as \[ C(T^2/mV_s^4)^{1/2} - 0 \] in this limit (assuming that \( C \) is held constant). Thus, Instantaneous shielding disappears in the very limit where Debye and Dynamic shielding work best. (Note that these conclusions have only been established for square-test wells in infinite-length plasmas, and ignore the effects of any instabilities due to the cusps in the distribution functions, and these effects may be important in many realistic situations.)

The results of this section assume that a steady state has been set up, i.e. that phase mixing has had time to occur \((t > 1/\omega_p)\), and that the trapped particles have made several bounces \((t > 1/\omega_p)\). Numerous authors\(^{15-17}\) have analyzed the linear stage of Instantaneous shielding for small initial perturbations \((\varepsilon \Phi < kT)\). As the method used by these authors apply only to the linear stage of the process, the results are only formally correct before trapped particles bounce, i.e. for times satisfying \(1/\omega_p < t < 1/\omega_p\).

C. Two and three-dimensional adiabatic theory

When electrons are no longer constrained to move in one dimension, their orbits will bend towards a positive test charge. If the test charge is two-dimensional, e.g. a line charge, the free-electron orbit bending near the test charge exactly compensates for the velocity increase there, and the net density of free electrons near the test line charge does not change. Consequently, previous authors\(^{18}\) have claimed that test line charges are neither shielded nor anti-shielded by collisionless plasmas. Were this true, however, the line charge's logarithmic potential would eventually trap all the electrons and there would be no free electrons.

The exact 2-D response of the plasma is \( n_2 = n_0 (1 + \Phi(r)) \), where \( \Phi(r) \) is a radially symmetric, two-dimensional potential. The first term represents the density of the untrapped electrons, and is found by using the constancy of the total energy to solve Vlasov's equation. The second term represents the density of the trapped electrons, and is found using the fact that only electrons with zero initial velocity are trapped if the potential \( \Phi(r) \) is turned on adiabatically. The exact self-consistent solution of the potential equation is \( \Phi(r) = \Phi_0(\kappa_0/\lambda) \), where \( \kappa_0 \) is a modified Bessel function.\(^{19}\) In the limit of large \( r \), \( \kappa_0(\lambda/\kappa) \) falls off exponentially with scale length \( \lambda \).

\[ n_s(\Phi) = n_0 \{ \exp(\Phi) + \text{erf}(\sqrt{\Phi})[1 - \exp(\Phi)] \} = n_0(1 + \Phi + \ldots). \quad (13) \]
In three dimensions, the electron orbits are sufficiently strongly bent towards a point test charge that the net electron density increases in the vicinity of the charge, despite the electrons' increased velocity. Thus, a point charge is collisionlessly shielded by the free electrons alone. The net electron density is 

$$n_{e0} = n_0 \exp(\Phi) \text{erfc}(\sqrt{\Phi}) + 2 \sqrt{\Phi} / \pi + 4 \sqrt{\pi} (\Phi)^{3/2},$$

(14)

where the free-electron contribution is given by the first two terms,\(^\text{2}\) and the trapped-electron contribution is given by the remaining term. While unimportant for very small values of \(\Phi\), the trapped electrons become the dominant shielding factor for \(\Phi > 2\).

D. Finite length, 1-D adiabatic theory

The above results assume that the background plasma is both Maxwellian and infinite in longitudinal extent. In many cases, however, including the pure-electron plasma experiments reported here, the plasma is of finite size. The initial Maxwellian distribution of such plasmas may be substantially modified by the creation of the test well. In these circumstances the plasma response is most readily calculated using the bounce adiabatic invariant, \(J = q_0 dv/da\). When the test potential is square, the invariance of \(J\) implies

$$L_{in} + L_{out} v_{in} = L_{in} v_{out} + L_{out} v_{out},$$

(15)

where \(v_{out}\) is the velocity of an electron before the test well is created, \(v_{in} (v_{out})\) is the velocity of this same electron inside (outside) the test well after the test well is created, and \(L_{out}\) (\(L_{out}\)) is the plasma length inside (outside) the test well. Using energy conservation to relate the outside velocity to the inside velocity \((v_{out}^2 = v_{in}^2 + 2\Phi)\) and Eq. (15), we can construct a function which gives the initial velocity \(v_{in}\) as a function of the final velocity inside or outside the well, e.g. \(v_{in} = v_{out} (v_{in}, \Phi)\). Since \(v_{in}\) is a constant of the motion, the final distribution functions can be expressed in terms of the initial distribution function \(f_{out}(v_{in})\), for example,

$$f_{in}(v_{in}) = f_{out}(v_{in}, \Phi)\]$$

As the test potential well deepens, the bounce invariant requires that electrons outside the well slow lose energy. Some electrons will eventually reach \(v_{out} = 0\), indicating that these electrons are trapped. We can still use \(f_{out}(v_{in}) = f_{out}(v_{in}, \Phi)\) so long as we define \(v_{in}\) by

$$L_{in} + L_{out} v_{in} = L_{in} v_{in}$$

(16)

rather than Eq. (15).

Typical distribution functions for a positive test well are shown in Fig. 1. The densities inside and outside the well are found by integrating the appropriate distribution functions over \(v\); the integral for the trapped component can be performed analytically,

$$n_{e}(\Phi) = n_{0} (1 + L_{out}/L_{in}) \text{erfc}(\sqrt{\Phi}/(1 + L_{out}/L_{in})).$$

(17)

but the integrals for the free electrons must be computed numerically. Once the total density \(n_{e}(\Phi)\) is determined, the self-consistent solution to the finite-length shielding problem is readily obtained as outlined above.

Negative potential well can be similarly analyzed by swapping the values of \(L_{in}\) and \(L_{out}\), inverting the sign of the applied potential, and calculating the total density in the new \(L_{out}\) region. The finite size case is substantially more complicated than the infinite case, and there is no closed form solution for the density like Eq. (7).

III. EXPERIMENT

The above theory was verified by pure-electron plasma experiments conducted in a Penning-Malmberg trap. Detailed descriptions of Penning-Malmberg traps can be found in the literature.\(^\text{23,24}\) In our trap the plasma forms a cylinder aligned along the common axis of a series of eleven collimated, cylindrical, 1.905 cm radius electrodes (see Fig. 5). The electrodes are biased to create an electrostatic well, thereby providing axial confinement. A strong axial magnetic field (1800 G) provides radial confinement, and also ensures that the electron motion is one dimensional. The plasma is generated by thermionic emission from a hot tungsten filament. The plasma is heated by applying a broadband noise signal of variable amplitude to one of the confining cylinders, and the plasma temperature is measured using the standard dynamic-evaporation technique.\(^\text{25}\) The plasma’s radial profile is determined by radially scanning a pinhole across the plasma while the plasma is being dumped. The typical plasma radius is approximately 1 cm, the typical density ranges from \(n_{e} \approx 1.2 \times 10^{10} \) to \(4 \times 10^{17} \) cm\(^{-3}\), and the plasma temperature ranges from 1 to 20 eV.

We generate a test well by appropriately biasing the trap electrodes, and determine the plasma response by finding the total charge contained within the test well. This total charge equals the image charge on the test well electrode, and is measured by integrating the current which flows onto the test well electrode. (To reduce undesirable coupling, we actually create the test well by leaving the well electrodes at ground and reverse biasing the remaining trap electrodes. The electrostatic well potential thus created is identical to that created by the more straightforward scheme of biasing the well electrodes themselves. The coupling is further minimized by guard electrodes.)

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Typically the test well is created adiabatically in a pre-existing plasma. Figure 6 shows the resulting plasma response as a function of the well potential. As predicted by the theory, the charge in the test well increases as the test well depth is increased, thereby shielding the applied potential. The charge increase is due to charge trapped in the well. By allowing the free charge to escape and then measuring the remaining charge, we can approximately determine the amount of the trapped charge (see Fig. 6). Because the self-consistent condition changes to

$$\Phi = \Phi_{s} + C[\eta(\Phi)/n_{0} - 1],$$

(18)

this trapped charge is not precisely the same as the trapped charge which exists before the free charge is allowed to escape. Consequently, the trapped charge shown in the figure does not follow the same scaling ($\sqrt{\Phi}$) as the initial trapped charge.

If the test well is created before the plasma is introduced into the well region (i.e. non-adiabatically), the sudden plasma expansion forms a phase-space cavity. The resultant cavities are large and long-lived. As shown in Fig. 6, phase space holes with density modulations as large as approximately 50% are observed. The cavities appear to oscillate from one end of the trap to the other, last for several milliseconds (thousands of plasma periods and axial bounces), and increase in size for well potentials less than 10 volts. Instabilities set in at about 10 V, and prevent us from measuring the response for well voltages greater than 20 V. The data is in rough agreement with a PIC simulation.\textsuperscript{25}

Figure 7 shows the total well charge vs. the plasma temperature for several well potentials. The positive potentials form well charges for the electrons and are shielded by resultant excess of electrons. The negative potentials form potential hills for the electrons and are shielded by the resultant deficit of electrons. The square points in the figure show the Dynamic response; the measurements are taken in a time much less than the collision time. The round points show the Debye response; the well is created over a length of time of $\sim 1$ s, far longer than the collision time for the plasma. As predicted by the theory, the Debye response is stronger than the Dynamic response for both hills and wells.

Figure 8 shows the trapped charge remaining after the free particles have escaped as a function of the plasma temperature. Similar to Fig. 6, the change in the self-consistent equation makes this change somewhat different than the trapped charge before the free particles escape. As expected, more charge is trapped in the Debye case than in the Dynamic case.

Finally, Fig. 9 contrasts the distribution function found inside a ten volt well with the initial, presumably Maxwellian distribution function found before the application of the well. The distribution functions were found by first using an electrode to split the plasma into inside and outside (the...
FIG. 9. Experimental distribution functions. Shown are the experimental data (solid lines) before and after the application of a +10 V well, the (a) best fit Maxwellian ($T=3.8$ eV) to the initial plasma, the (b) calculated "Dynamic" distribution in the well, and (c) the calculated Maxwellian ($T=5.2$ eV) to which the experimental data should relax due to collisions. Here $L_w/L_{	ext{sat}}=0.77$.

well) regions, and then measuring the number of electrons that escape as a function of the end confinement-barrier energy $E$. We avoid the confounding effects of the plasma space charge by using a very tenuous plasma ($C=0.26$). This is a difficult measurement, and noise limits us to presenting only integrated data, $n(E) = 2n_0\int_{E_0}^{E} \frac{\pi\rho_0}{4} (\frac{m}{2\pi kT})^{3/2} v^2 dv$. Nonetheless, the distribution function inside the well is clearly non-Maxwellian.

IV. DISCUSSION

We have shown that 1-D collisionless shielding relies on trapping, and without trapping, a plasma anti-shields a positive test charge. Since the electrons in highly magnetized plasmas respond one dimensionally, 1-D shielding is common. Further, the standard 2-D and 3-D derivations require modification to model trapping correctly, and the plasma response in 2-D is qualitatively different than reported elsewhere. The distribution functions found in collisionless (Dynamic) shielding are non-Maxwellian, and the Dynamic response can be far weaker than the equilibrium Debye response in the nonlinear regime. We have experimentally demonstrated both shielding and anti-shielding, and our results are in excellent agreement with the predictions of 1-D Dynamic and Debye theory.

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6. One-dimensional plasmas are defined to be plasmas in which electrons are restricted to one-dimensional motion, either by geometry or by a strong linear magnetic field.
12. Typically, these sources employ a Maxwell-Boltzmann distribution $e^{-\delta\mu/kT}e^{-\mu^2/2kT}$ without noting that collisions are required to set up this distribution after the application of the test potential. While it is possible to casually derive this distribution from the collisionless Vlasov equation by starting with an initial Maxwellian distribution $e^{-\mu^2/2kT}$, the proper, initial-value solution of Vlasov's equation is different from the Maxwell-Boltzmann distribution (see Refs. 2 and 15-17).
19. The simulation was written by D. Durkin, University of California, Berkeley.
22. Two-dimensional shielded potentials proportional to $K_0(r/A_0)$ have been reported before based on linearization of the Boltzmann relation (see B. Abraham-Shrauner, *Physica* 43, 95 (1969)). However this expression would not have been found had the derivations been continued to higher order. Only Dynamic shielding gives this expression exactly.