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NOTE

Application of Contour Dynamics to Systems with Cylindrical Boundaries

1. INTRODUCTION

Contour dynamics (CD) is a widely used method in fluid mechanics for studies of two-dimensional vortical flows of incompressible, inviscid fluids. It was originally developed by Deem and Zabusky [1, 2] for unbounded flows. A review of the method and numerical implementations can be found in Ref. [3]. CD is based on the simplification present for flows with piecewise-constant distributions of vorticity in which the entire fluid motion can be determined by the locations of vortex boundaries. Under these conditions, the problem is effectively reduced from two dimensions to one. Moreover, CD represents the vorticity distribution in a smoother (and thus more realistic) manner than other techniques. A refinement of the method, named "contour surgery" [4], was also developed to resolve problems associated with the development of fine spatial scales, thereby making long time calculations possible.

Contour dynamics cannot be used in its standard form for bounded systems, where the normal component of velocity (or the value of stream function) has to satisfy a boundary condition. Some work on application of CD to different geometries was done in the past: the half-plane was considered in Ref. [5]; CD on the surface of a sphere can be found in Ref. [4]. Nevertheless, no ready algorithm can be found in the literature for bounded planar surfaces. In this paper, a straightforward CD method is presented that can be used for systems with cylindrical boundaries and can be generalized to other geometries. The free-space part of the stream function (ψ_f) is calculated using standard CD, while the boundary-induced image contribution (ψ_i) is computed as a solution of the Laplace's equation which satisfies the condition $\psi_i|_{C_b} = -\psi_f|_{C_b}$, where C_b is the system boundary.

As an illustration, the method is applied to the problem of finding the static equilibria of a nonneutral plasma column subjected to an externally applied boundary potential. The system is comprised of a long electron plasma column which is confined radially by a uniform axial magnetic field and is bounded in space by a cylindrical wall. Fluid-like, two-dimensional (2D) motion occurs in the plane perpendicular to the magnetic field. The dynamics is governed by the continuity equation for the electron fluid density (n), along with the velocity determined by $\mathbf{E} \times \mathbf{B}$ drift and Poisson's equation for the electrostatic

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potential. As is well known [6], these 2D drift–Poisson equations are isomorphic to Euler's equations for a constant density fluid with the charge density playing the role of the vorticity.

2. CONTOUR DYNAMICS METHOD WITH CYLINDRICAL BOUNDARIES

We start with a short statement of CD for unbounded flows [1, 2]. The velocity of the 2D flow at any point can be expressed in terms of the stream function as $\mathbf{v}(x, y) = \nabla\psi(x, y) \times \hat{\mathbf{e}}_\mathbf{z}$. The stream function ψ satisfies Poisson's equation, $\nabla^2\psi = -\omega$, where ω is the vorticity. It is assumed that the vorticity is represented by a set of N_c piecewise-constant values ω_k, $k = 1 \ldots N_c$, in regions $\mathcal{R}_k$ bounded by contours C_k. Using the 2D Green function to solve Poisson's equation, the free-space stream function can be written as

$$\psi_f(x, y) = -\frac{1}{2\pi}\int\int dx' dy' \omega(x', y') \ln|\mathbf{r} - \mathbf{r}'|. \tag{1}$$

Using Stokes's theorem, the area integral in Eq. (1) can be converted into a line integral over the boundary

$$\psi_f(x, y) = \sum_{k=1}^{N_c} \frac{\omega_k}{4\pi}\left(A_k - \oint_{C_k} \ln|\mathbf{r} - \mathbf{r}'|[(x' - x)\, dy' - (y' - y)\, dx']\right), \tag{2}$$

where A_k is the area of the region $\mathcal{R}_k$. The velocity field can also be written in terms of contour integrals, namely

$$\mathbf{v}_f(x, y) = -\sum_{k=1}^{N_c} \frac{\omega_k}{2\pi}\oint_{C_k} \ln|\mathbf{r} - \mathbf{r}'|\, d\mathbf{x}'. \tag{3}$$

Note that if the vorticity outside the vortex patch $\mathcal{R}_k$ is not equal to zero, then ω_k in Eqs. (2) and (3) should be replaced by the jump of vorticity across the boundary C_k (the inside vorticity ω_k minus the outside vorticity). Thus, the problem of time evolution of regions with constant vorticity is now reduced to the evolution of boundaries of C_k.

We now turn our attention to bounded systems. In the presence of boundaries, C_b, the stream function is no longer equal simply to ψ_f, but can be expressed as

$$\psi = \psi_f + \psi_i + \psi_{ex}, \tag{4}$$

where ψ_i is the stream function due to the boundary-induced image of the vortex distribution and ψ_{ex} corresponds to the external stream function which satisfies the prescribed boundary condition on C_b.

The image stream function can be found in a number of different ways. In certain geometries (e.g., a half-space) ψ_i can be written, using the corresponding Green function, as an area integral and reduced to the line integral analytically using Stokes's theorem. This method does not work in all geometries, and the Green function cannot be found analytically for complicated boundaries. We take a different approach. The image contribution is constructed as a solution to the Laplace equation, $\nabla^2\psi_i = 0$, subject to the appropriate boundary condition $\psi_i|_{C_b} = -\psi_f|_{C_b}$. This guarantees that the total stream function satisfies the boundary condition. The values $\psi_f|_{C_b}$ are calculated by computing Eq. (2) along the boundary C_b.

Consider a system with a cylindrical wall of radius R. The general image stream function can be written for $r \leq R$ as

$$\psi_i(r,\theta) = \sum_{l=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|l|} D_l\, e^{-il\theta}, \tag{5}$$

with $D_{-l} = D_l^*$ to provide a real function ψ_i. In practice, the sum in Eq. (5) is truncated at some maximum value of $l_{\max} = N_l$. The set of coefficients D_l is found from the boundary condition $\psi_i(R,\theta) = -\psi_f(R,\theta)$. Using Eq. (2) for ψ_f, the following expression for the coefficient D_l ($l \neq 0$) is obtained,

$$D_l = \sum_{k=1}^{N_c} \frac{\omega_k}{8\pi^2} \int_0^{2\pi} d\theta e^{il\theta} \oint_{C_k} \ln|\mathbf{R} - \mathbf{r}'|[(x' - R\cos\theta)\, dy' - (y' - R\sin\theta)\, dx'], \tag{6}$$

with $\mathbf{R} = (R\cos\theta, R\sin\theta)$. Note that the inner integral is one-dimensional and thus can be numerically evaluated quickly. The outer integral is a fourier transform and can likewise be evaluated quickly. Thus, this method of evaluating the image stream function is very efficient.

A nonzero, external boundary condition can be taken into account in a similar manner. The external field ψ_{ex} is written as a solution of Laplace's equation that satisfies the prescribed boundary condition at C_b.

Velocities, needed for the calculation of the time evolution of the contours, can be evaluated using Eq. (3) for the free space part v_f and by simple differentiation of resulting expressions for ψ_i (e.g., Eq. (5)) and ψ_{ex}.

3. APPLICATIONS TO ASYMMETRIC PLASMA EQUILIBRIA

As an illustration of the method, we consider a pure-electron plasma column confined in a cylindrical Penning–Malmberg trap [7]. As discussed in the Introduction, the equations of motion for the magnetically confined electron column are identical to those governing a 2D inviscid incompressible fluid [6] with the density playing the role of vorticity ($\omega \to 4\pi\, enc/B$) and electrostatic potential replacing the stream function ($\psi \to -c\phi/B$).

We will use CD (more precisely, contour statics) to find the asymmetric equilibria of the column in the presence of external potential $V(\theta)$ applied to the confining wall. In the absence of applied asymmetries $V(\theta) = 0$, the column assumes a perfectly cylindrical equilibrium shape. When azimuthally asymmetric potentials are applied to the wall, the plasma deforms into an asymmetric equilibrium. These equilibria have been observed experimentally [8]. In a previous theoretical work [9], the static shapes were studied analytically for small applied potentials and numerically for arbitrary potentials. There, the equilibria were obtained dynamically (by following the evolution of the initially circular plasma as the wall potential is slowly changed) using a time consuming and inaccurate vortex-in-cell simulation. These equilibrium shapes can be found much more easily using the technique presented in this paper.

For a given external potential applied to the wall there might be a few, one, or no equilibria. Note that these equilibria are not necessarily stable. To find the static shape, we first note that the guiding centers of the plasma electrons drift along equipotential contours. Therefore, the plasma shape is stationary if its boundary coincides with an equipotential contour [9]. Mathematically, the condition for the plasma to be in equilibrium can be written as

$$\phi(r_p, \theta) = \text{const}, \tag{7}$$

where $r_p = r_p(\theta)$ specifies the plasma boundary. Representing the contour in the coordinate system centered inside plasma at x_c, y_c as

$$x_p(\theta) = x_c + \xi(\varphi)\cos\varphi, \quad y_p(\theta) = y_c + \xi(\varphi)\sin\varphi, \tag{8}$$

and using Eqs. (2) and (5), this equilibrium condition can be written explicitly as

$$\begin{aligned} en \int_0^{2\pi} \ln q(\varphi, \varphi') \left[\xi'^2 - \xi'\xi\cos(\varphi' - \varphi) - \frac{\partial \xi'}{\partial \varphi'}\xi \sin(\varphi' - \varphi)\right] d\varphi' \\ + \sum_{l=-\infty}^{\infty} \left(\frac{r_p}{R}\right)^{|l|} (D_l + \tilde{V}_l)\, e^{-il\theta} = \text{const}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \xi &= \xi(\varphi), \quad \xi' = \xi(\varphi'), \quad r_c^2 = x_c^2 + y_c^2 \\ q^2 &= \xi^2 + \xi'^2 - 2\xi'\xi\cos(\varphi' - \varphi), \\ r_p^2 &= r_c^2 + \xi^2 + 2r_c\xi\cos(\varphi - \theta_c), \\ \tan\theta &= (y_c + \xi\sin\varphi)/(x_c + \xi\cos\varphi), \\ \tan\theta_c &= y_c/x_c. \end{aligned} \tag{10}$$

$\tilde{V}_l$ is a Fourier transform of the wall potential $V(\theta)$, and D_l can be calculated using Eq. (6):

$$D_l = -\frac{en}{2\pi}\int_0^{2\pi} d\theta e^{il\theta} \int_0^{2\pi} d\varphi' \ln q(\varphi, \varphi')\left[\xi'^2 - \xi' R\cos(\varphi' - \varphi) - \frac{\partial \xi'}{\partial \varphi'} R \sin(\varphi' - \varphi)\right] d\varphi'. \tag{11}$$

Equation (9) is a closed-form integro-differential equation for the plasma equilibrium shape with D_l given by Eq. (11).

The plasma boundary is discretized into N points, and Eq. (9) is solved numerically. For simplicity the wall potential is assumed to be an even function in y (but also see [10]), and the equilibrium centered on the x-axis at x_c is sought. Positions of the boundary points are then represented in the coordinate system associated with plasma as given by Eq. (8) with $\varphi_i = 2\pi i/N$. The curve between two adjacent points is interpolated by cubic splines $\mathbf{r} = \mathbf{r}_i + s\mathbf{t}_i + \eta(s)\mathbf{n_i}$, where $\mathbf{t}_i$ and $\mathbf{n_i}$ are the tangent and normal to the curve, respectively, and s is the fractional distance between the nodes. Integrals from Eqs. (9) and (11) are calculated analytically to first order in η (for a discussion of this technique see Ref. [4]). The equilibrium condition, Eq. (7), written at the N boundary points results in N nonlinear algebraic equations for $N+1$ unknowns: x_c, ξ_i $(i = 1, N)$. The system is supplemented with the equation for the area of plasma. To find an equilibrium corresponding to a given wall potential the initial guess is used as a starting point and the equations are solved using

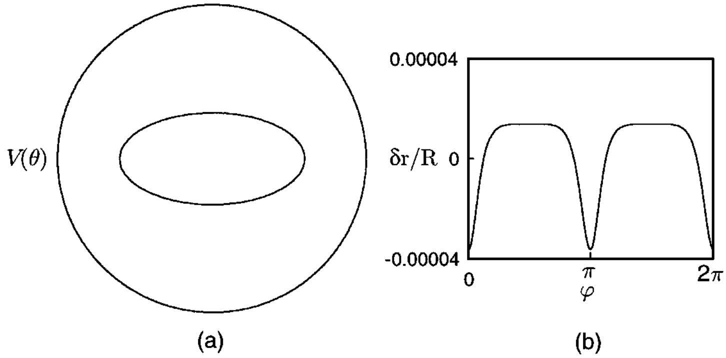

FIG. 1. (a) Elliptical equilibrium. (b) Absolute error between the numeric and analytic solutions.

Broyden's method [11]. The error criterion is defined in the following way. The average plasma surface potential is computed at every iteration as

$$\bar{\phi} = \frac{1}{N}\sum_{i=1}^{N} \phi(r_p(\varphi_i), \varphi_i). \tag{12}$$

The run is terminated if

$$\sup_{i\in[1,N]} |\phi(r_p(\varphi_i), \varphi_i) - \bar{\phi}| < \varepsilon, \tag{13}$$

where ε is the desired accuracy. The solution found by numerical integration corresponds to the equilibrium plasma shape of a specified area and a given potential applied to the confining wall.

To test the performance of the procedure, numerical integration was performed for a number of equilibria with known analytical shapes [12]. Potentials corresponding to off-center circles and ellipses were used. As an illustration of the agreement between analytical and numerical results, consider the following example. The wall potential is taken such that the known analytical equilibrium is an ellipse with $a = 0.6R$, $b = 0.3R$, where a and b are the major and minor axes of the ellipse, respectively. As this ellipse is quite distorted from a circle, it is a good test of the robustness of the method. The calculation is performed using $N = 256$ points to represent the contour, and the sum in Eq. (5) is truncated at $N_l = 128$. The initial guess was taken to be a circle, and the desired accuracy of $\varepsilon = 10^{-7}$ was achieved in 14 iterations. The result of the numerical integration and the absolute error between the analytical and numerical solutions $\delta r = r^{num} - r^{anal}$ can be seen in Figs. 1a and b, respectively.

In the next example, which cannot be solved analytically, a constant normalized potential $V^* = V/4\pi e n_0 R^2$ is applied to two 90° sectors with the rest of the cylindrical wall being grounded. The area of the plasma is taken to be $A_p/\pi R^2 = 0.25$. Equilibria for three different potentials $V^* = 0$, $V^* = -0.08$, and $V^* = -0.16$ are presented in Fig. 2. The well-known, but peculiar, attraction of the negatively charged plasma to negative voltages [8] can be seen, as well as a severe deformation of the plasma shape from cylindrical for large asymmetries.

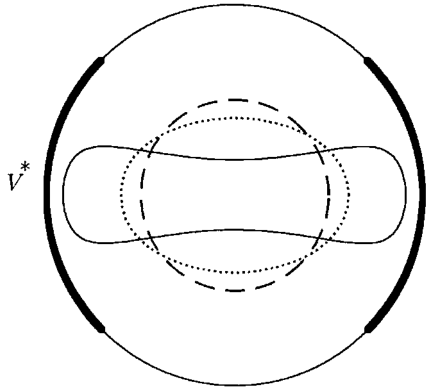

FIG. 2. Equilibria determined numerically for $V^* = 0$ (- - -), $V^* = -0.08$ (·····), $V^* = -0.16$ (—) applied to opposing 90° arcs (drawn in as the heavy lines).

4. CONCLUSIONS

The CD method, which was originally developed for unbounded flows, has been extended to take into account a cylindrical boundary conditions and external fields. The method was tested for a static problem of plasma equilibria in a presence of an external asymmetric potential. The technique can be also generalized to account for boundaries other than cylindrical.

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