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NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH
Section A

Computationally efficient spectral analysis of an FEL oscillator using a Green function analysis

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Abstract

Previous work on a Green function approach to the evolution of the small gain linear FEL oscillator assumed a long electron pulse. Here this restriction is eliminated and a new expression for the complex amplitude of the optical field at the *n*th pass is obtained. The fully causal formalism includes arbitrary temporal profiles of the electron and optical beams and cavity detuning and losses. In our formalism, the solution for the field at the *n*th pass requires exponentiating a single matrix rather than sequentially evaluating the field at each pass. Various numerical studies are presented.

1. Introduction

The spectral properties of free-electron laser (FEL) oscillators are of critical importance to many applications and have been the subject of numerous investigations. Most of these studies have been made with numerical simulations of the coupled particle–field equations. Here we restrict our analysis to the linear regime, where analytical techniques can be used. There has been a large body of work regarding the FEL as a (time-independent) dielectric medium, but this has been confined mainly to linear gain analysis and optical guiding. The analysis of the linear time-dependent oscillator FEL has received less attention. Colson [1] formulated the Compton FEL interaction as an integral equation for the complex field amplitude. Growth from noise (SASE) in the single pass FEL was examined by Kim [2] and by Bonifacio and co-workers [3]. A Green function approach was used in Ref. [3], and in studies of spiking in oscillators by Jerby et al. [4] and Shvets and Wurtele [5]. Supermode theory by Dattoli et al. [6] and Elleaume [7] addressed the pulse evolution in the FEL oscillator, but required a noncausal approximation. The causal multi-pass Green function for a low gain oscillator was found in Ref. [5]. In general, the above works assumed an electron beam that was infinite, slowly varying on the time scale of the optical pulse, or had a step-function profile. A formulation of FEL oscillator pulse evolution by Fourier transform techniques and matrix multiplication was developed by Tang and co-workers [8]. In this paper we extend the analysis of Ref. [5] to include arbitrary electron beam profiles.

The Green function approach to the FEL is conceptually appealing. Because of slippage, each slice of the optical pulse leaves a "wake" on the electron beam and experiences a (complex) gain as it propagates. The general form for a single pass Green function would then be expected to depend parametrically on interaction length. The parameters of the electron beam or complex field amplitude can be time-dependent, causing different temporal slices of the radiation pulse to experience unequal gain, which leads to spectral changes. The slippage allows a finite radiation pulse propagating on a uniform electron beam to experience frequency shifting in regions where the radiation pulse varies. In another paper in these proceedings we derive the single pass Green function for the Raman FEL. In an FEL oscillator there is gain, slippage, cavity detuning and losses and multiple passes, so that any analysis is substantially more cumbersome than that used, for example, in studies of the evolution of light in a time-dependent plasma.

The analysis here is fully causal, valid in the low gain regime, and is computationally efficient compared to iterating the single pass Green function. The formulation includes finite electron and optical pulses and cavity detuning and losses. It can be used to examine numerous time-dependent linear phenomena in oscillators, and, in the future, may be (phenomenologically) extended to include nonlinear saturation.

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2. Theoretical model

This paper follows the formalism of Ref. [1], where the normalized independent variables are

$$\bar{s}=\frac{2\gamma_{\|}^{2}c}{N_{w}\lambda_{w}}\left(t-\frac{z}{v_{g}}\right), \tag{1}$$

$$\bar{z}=z/(N_{w}\lambda_{w}). \tag{2}$$

These variables are chosen so that $\bar{s}$ propagates with the optical pulse and is measured in units of the slippage length, and interaction length z is normalized to the wiggler length; thus $\mathrm{d}\bar{s}/\mathrm{d}\bar{z}=1$. The free-streaming phase of a particle in the ponderomotive wave is given by

$$\theta_{0}=2\pi N_{w}(\bar{z}-\bar{s})+\bar{s}\left(k_{w}-k_{z}/2\gamma_{\|}^{2}\right)N_{w}\lambda_{w}, \tag{3}$$

where the detuning normalized by the wiggler length is y_0. Also, N_w is the number of wiggler periods, λ_w is the wiggler period, $\gamma_{\|}$ is the electron relativistic factor calculated from the parallel velocity, c is the speed of light, and v_g is the group velocity of light (equal to c in the numerical example below).

Within the context of a cold beam model prior to saturation, the FEL evolution equations for an arbitrary gain during a single pass through the wiggler can be written as:

$$\frac{\partial\hat{a}}{\partial\bar{z}}=\frac{2\pi\mathrm{i}a_{w}L_{w}r_{e}\delta n}{\gamma k}\exp(-\mathrm{i}\theta), \tag{4}$$

$$\frac{\mathrm{d}^{2}\theta}{\mathrm{d}\bar{z}^{2}}=\mathrm{i}\frac{8\pi^{2}N_{w}^{2}a_{w}}{1+a_{w}^{2}}\hat{a}[\bar{s}(\bar{z}),\bar{z}]\exp(\mathrm{i}\theta), \tag{5}$$

$$y_{0}=\frac{\mathrm{d}\theta}{\mathrm{d}\bar{z}}(\bar{z}=0),\quad \theta_{0}=\theta(\bar{z}=0). \tag{6}$$

In this notation the normalized slowly varying complex amplitude of the field is $\hat{a}(\bar{s},\bar{z})$, the detuning is y_0, r_e is the classical electron radius, the Lagrangian phase variable is θ, the wiggler amplitude is a_w, $L_w=N_w\lambda_w$, $k=2\pi/\lambda$ is the radiation wavenumber, and the density perturbation δn can be expressed in terms of the phase as:

$$\delta n=-n_{0}(\bar{s}-\bar{z})\left(\frac{\partial\theta(\theta_{0},\bar{z})}{\partial\theta_{0}}-1\right), \tag{7}$$

where n_0 is the beam density. Note that $t-z/v_z=(N_w\lambda_w/c)(\bar{s}-\bar{z})$. In linear theory, the phase can be expanded as:

$$\theta=\theta_{0}+y_{0}\bar{z}+\mathrm{i}\frac{8\pi^{2}N_{w}^{2}a_{w}}{1+a_{w}^{2}}\int_{0}^{\bar{z}}\mathrm{d}\bar{z}'\int_{0}^{\bar{z}'}\mathrm{d}\bar{z}''\ \hat{a}(\bar{s}_{0}+\bar{z}'',\bar{z}'')\times\exp[\mathrm{i}(y_{0}\bar{z}''+\theta_{0})], \tag{8}$$

and an integral equation for the field can be obtained (see Ref. [6]):

$$\frac{\partial\hat{a}(\bar{z},\bar{s})}{\partial\bar{z}}=ij_{c}(\bar{s}-\bar{z})\exp(-\mathrm{i}y_{0}\bar{z})\times\int_{0}^{\bar{z}}\mathrm{d}\bar{z}'\int_{0}^{\bar{z}'}\mathrm{d}\bar{z}''\ \hat{a}(\bar{s}-\bar{z}+\bar{z}'',\bar{z}'')\exp(\mathrm{i}y_{0}\bar{z}''), \tag{9}$$

where $j_c=4\pi a_w^2N_w^3\lambda_w^2I/(\gamma^3r_b^2I_A)$, $I(\bar{s}-\bar{z})$ is the beam current, r_b is the beam radius, and $I_A=17$ kA. Integration over the wiggler length yields a difference equation for the multi-pass gain:

$$\Delta a(s;n)=\mathrm{i}\int_{0}^{1}\mathrm{d}\bar{z}\int_{0}^{\bar{z}}\mathrm{d}\bar{z}'\int_{0}^{\bar{z}'}\mathrm{d}\bar{z}''j_{c}(\bar{s}-\bar{z})\times\hat{a}(\bar{s}-\bar{z}+\bar{z}'',\bar{z}'';n)\times\exp[\mathrm{i}y_{0}(\bar{z}''-\bar{z})]. \tag{10}$$

In the limit that the gain is small, the variation of the field with respect to interaction length can be ignored in the integral and a differential equation for the multi-pass gain can be obtained:

$$\frac{\partial a(s;n)}{\partial n}=\mathrm{i}\int_{0}^{1}\mathrm{d}\bar{z}\int_{0}^{\bar{z}}\mathrm{d}\bar{z}'\int_{0}^{\bar{z}'}\mathrm{d}\bar{z}''j_{c}(\bar{s}-\bar{z})\hat{a}(\bar{s}-\bar{z}+\bar{z}'';n)\times\exp[\mathrm{i}y_{0}(\bar{z}''-\bar{z})]. \tag{11}$$

The calculation proceeds most readily by a Fourier transform in $\bar{s}$:

$$\hat{a}(\omega;n)=\int_{-\infty}^{\infty}\mathrm{d}s\ \exp(\mathrm{i}\omega\bar{s})\hat{a}(\bar{s},n), \tag{12}$$

$$J(\omega)=\int_{-\infty}^{\infty}\mathrm{d}s\ \exp(\mathrm{i}\omega\bar{s})j_{c}(\bar{s}-\bar{z},n). \tag{13}$$

Then

$$\frac{\partial\hat{a}(\omega;n)}{\partial n}+\left(\mathrm{i}\omega\Delta+\frac{1}{2Q}\right)\hat{a}(\omega;n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega'\ J(\omega-\omega')G(\omega,\omega')\hat{a}(\omega';n), \tag{14}$$

where Δ is twice the cavity detuning normalized to the slippage length ($\Delta=2\delta L_c/(N_w\lambda_s)$), Q is the cavity loss, and the Green function G is

$$G(\omega,\omega')=i\int_{0}^{1}\mathrm{d}\bar{z}\int_{0}^{\bar{z}}\mathrm{d}\bar{z}'\int_{0}^{\bar{z}'}\mathrm{d}\bar{z}''\ \exp[\mathrm{i}(\omega-\omega')\bar{z}'']\times\exp[\mathrm{i}(y_{0}-\omega)(\bar{z}''-\bar{z})]. \tag{15}$$

The Green function is a product of exponentials and can be solved straightforwardly in closed form. The multi-pass evolution can now be solved for numerically.

3. Numerical method

We solve the equation by isolating the n-derivative and integrating and using a discrete Fourier transform. The result is a matrix equation for the nth pass field in terms of the initial field:

$$\begin{pmatrix} \hat{a}(\omega_1, n) \\ \cdot \\ \cdot \\ \cdot \\ \hat{a}(\omega_N, n) \end{pmatrix} = \exp(n\mathbf{T}) \begin{pmatrix} \hat{a}(\omega_1, 0) \\ \cdot \\ \cdot \\ \cdot \\ \hat{a}(\omega_N, 0) \end{pmatrix}. \quad (16)$$

Here the transport matrix $\mathbf{T}$ is readily calculated from the detuning, losses and Green function. The exponential of the matrix can be efficiently calculated by using $\exp(n\mathbf{T}) = (\mathbf{I} + \epsilon\mathbf{T} + O(\epsilon^2))^{n/\epsilon}$ and choosing n/ϵ to be a power of 2. Typically n/ϵ is chosen to be 1024 and the Fourier transform is taken with 256 bins.

4. Examples

In a previous study [5] of a low gain oscillator, the causal multi-pass Green function was found in the limit of a long electron bunch:

$$\hat{a}(\bar{s}; n) = \int_{\bar{s}-\bar{z}}^{\bar{s}} d\bar{s}'\ G_n(\bar{s} - \bar{s}')\hat{a}(\bar{s}'; 0),$$

$$G_n(\bar{s}) = \frac{1}{2\pi}\int_c d\omega\ \exp(-i\omega\bar{s} + nj_c - g(y_0 - \omega)),$$

where the low gain FEL spectral function is

$$g(y_0) = i\int_0^1 dz \int_0^{\bar{z}} d\bar{z}''\ \exp(iy_0(\bar{z}'' - \bar{z})).$$

These earlier results can be obtained from the present analysis in the limit of a long electron pulse ($J(\omega) = 2\pi\delta(\omega)$). The response to either a jump or a slow modulation of the input electron energy (after the radiation has developed with a spectrum centered around the maximum gain frequency) was investigated. After the energy jumps suddenly, the optical pulse is detuned, the FEL gain cannot compensate for the cavity losses, and the center frequency of the optical pulse shifts to the new peak gain value. Possibly, the frequency shift could occur by amplification of either random beam noise, synchrotron radiation, or a sideband signal. Our numerical and analytic work showed, rather, that the frequency can change through slippage on a finite optical pulse – and that the frequency adjustment requires neither beam nor field noise. For a flat-top pulse the correct frequency appears at the edges, where slippage is significant, and then diffuses into the bulk of the pulse. This was modeled with a diffusion equation:

$$\frac{\partial\hat{a}}{\partial n} = \frac{1}{2}\left(-j_c\left.\frac{d^2 g}{dy^2}\right|_{y=y_0}\right)\frac{\partial^2\hat{a}}{\partial\bar{s}^2} - \left(ij_c\left.\frac{dg}{dy}\right|_{y=y_0} - \Delta\right)\frac{\partial\hat{a}}{\partial\bar{s}} + \left(-\frac{1}{2Q} + j_c g(y_0)\right)\hat{a}.$$

It has long been known that the peak gain detuning is such that it compensates for the lethargy. This diffusion equation allows one to find an analytic expression for the optimal cavity detuning. From the diffusion equation, it is clear that, in the limit of a long electron beam, $-\mathrm{Im}(j_c\, dg/dy)$ evaluated at peak gain $y_0 = 2.6$ gives the lethargy. Note that this gives the criterion $\Delta = 0.7j_c$ (the slope of the imaginary part of the spectral function at peak gain is nearly 0.07, see also Ref. [6]). We examined this criterion for a few examples, corresponding to various values of the pulse length. In these studies, shown in Fig. 1, the cavity detuning was changed for three values of the pulse length. When slippage is unimportant the result agrees with our theoretical prediction ($-\mathrm{Im}(j_c\, dg/dy) =$

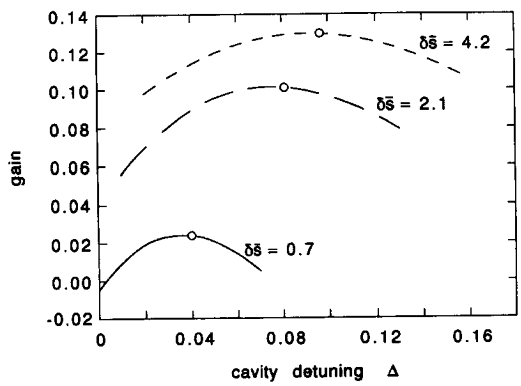

Fig. 1. Dependence of the linear gain on the cavity detuning is shown for normalized electron beam pulse lengths, $\delta\bar{s}$, of 0.7, 2.1 and 4.2, with the Colson parameter $j_c = 1.54$, and $Q = 10$. Detunings are varied between $\Delta = 0$ and $\Delta = 0.16$.

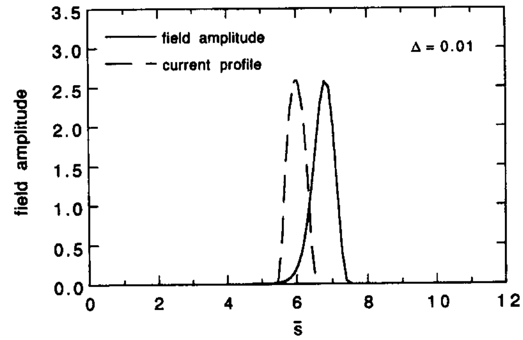

Fig. 2. The electron and optical pulse shapes after 100 passes for an electron beam which is 0.7 slippage lengths long. The electron pulse is shown, in the radiation frame, at the entrance to the wiggler; the electrons would slip by unity to the right in one pass. The cavity detuning is $\Delta = 0.01$, the Colson parameter $j_c = 1.54$, and $Q = 10$.

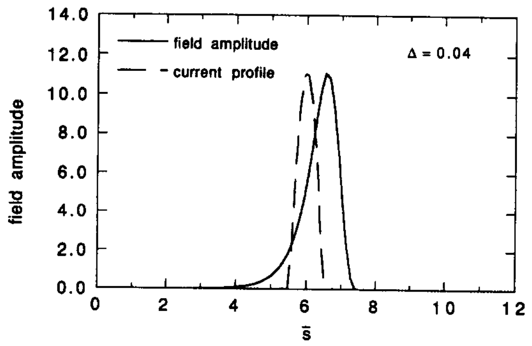

Fig. 3. The same as plot of pulse shapes as in Fig. 2 except the cavity detuning is $\Delta = 0.04$.

$\Delta = 0.096$) based on the diffusion model. As the pulse is shortened, slippage becomes more significant, and the gain is no longer given by the usual Colson parameter. As a measure of our understanding of the relation between the required cavity detuning and the peak gain, we estimated an effective Colson parameter required to reach the peak gain for each of the shorter pulses. Replacing j_c by its effective value resulted in good agreement with the theory, as shown by the circles that mark our theoretically estimated peak cavity detuning.

Shown in Figs. 2–4 are the electron and optical pulse shapes after 100 passes for an electron beam length equal to 0.7 slippage lengths. The electron pulse is shown, in the radiation frame, at the entrance to the wiggler; the electrons would slip by unity to the right in one pass. The cavity detuning is increased from 0.01 in Fig. 2 to 0.07 in Fig. 4.

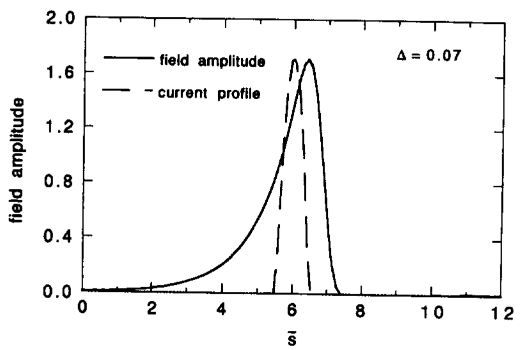

Fig. 4. The same as plot of pulse shapes as in Fig. 2 except the cavity detuning is $\Delta = 0.07$.

5. Discussion

This analysis of the FEL oscillator evolution has so far only been performed in the linear region and in a one-dimensional model. It is straightforward to include gratings, frequency-dependent loss factors, and dynamic cavity detuning. It is less straightforward, but of interest, to include transverse effects (diffraction and guiding) and nonlinear effects (saturation, nonlinear spiking, sidebands). We believe many spectral phenomena, previously suspected as occurring only through nonlinear interactions, can be driven by slippage induced frequency changes in the linear regime.

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