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Plasma shielding, Vlasov's equation, and the unperturbed-orbits technique

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Abstract

The common textbook derivation of collisionless plasma shielding is technically invalid. Trapped particles are crucial to shielding, but the trapped-particle distribution does not reach a steady state before the trapped particles bounce, thereby violating the unperturbed-orbit technique used in the derivation. However, physical arguments indicate that the final answer is nonetheless often correct. © 1999 Published by Elsevier Science B.V. All rights reserved.

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One dimensional, collisionless shielding of a test charge is critically dependent on the particles trapped in the vicinity of the test charge. If the number of trapped particles is calculated incorrectly, the degree and even the sign of the predicted shielding can be incorrect. We will show that by the time the trapped particles have settled down to a steady state, they will have bounced in the shielded potential well formed by the test charge, thereby violating the unperturbed-orbit technique used in textbook shielding derivations [1,2]. Thus these derivations are technically invalid.

In this Letter, we will assume a fully one-dimensional system. However, the same considerations apply to a system in which the potential is three-dimensional, but the particle's guiding center motion is constrained to straight lines: for instance, to a plasma in a strong magnetic field. In a fully three-dimensional system orbit bending is more important than particle trapping [3], and the issues raised here are unimportant.

We begin with a brief derivation of the shielding of a test sheet $\sigma\delta(x)$. Following Nicholson [1], we assume that the ions are immobile, and that the electron distribution function can be written as

$$f_e(x,v,t) = f_0(v) + f_1(x,v,t), \qquad (1)$$

where $f_0(v)$ is the unperturbed distribution and $f_1(x,v,t)$ is a perturbation. We will solve the linearized Vlasov equation,

$$\frac{\partial f_1(x,v,t)}{\partial t} + v\frac{\partial f_1(x,v,t)}{\partial x} = \frac{-e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f_0}{\partial v}, \qquad (2)$$

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in conjunction with Poisson's equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{e}{\epsilon_0}\int f_1(x,v,t)\,dv - \frac{\sigma}{\epsilon_0}\delta(x), \tag{3}$$

where e and m are the electron charge and mass. We proceed by taking the spatial Fourier transform and the temporal Laplace transform of Eqs. (2) and (3), yielding two new equations

$$-k^2\phi(k,\omega) = \frac{e}{\epsilon_0}\int f_1(k,v,\omega)\,dv + \frac{\sigma}{2\pi i\epsilon_0\omega},$$

$$i\omega f_1(k,v,\omega) - ikvf_1(k,v,\omega) = \frac{ike}{m}\phi(k,\omega)\frac{\partial f_0}{\partial v}. \tag{4}$$

Note that the Laplace transform implies that the test sheet is suddenly and fully turned on at $t=0$.

Nicholson solves this pair for the potential; we find it more informative to solve for the perturbed distribution,

$$f_1(k,v,\omega) = \frac{ie\sigma}{2\pi\epsilon_0 mk}\frac{1}{\epsilon(k,\omega)}\frac{1}{\omega}\frac{1}{(\omega-kv)}\frac{\partial f_0}{\partial v}, \tag{5}$$

where

$$\epsilon(k,\omega) = 1 - \frac{\omega_e^2}{k^2}\int\frac{df_0(u)/du}{u-\omega/k}du \tag{6}$$

is the Vlasov dielectric function. Here ω_e is the electron plasma frequency. No general expression exists for $\epsilon(k,\omega)$, but it can be simplified in certain limits. In particular, assuming that $f_0(v)$ is Maxwellian,

$$\epsilon(k,\omega) = 1 + \frac{1}{k^2\lambda_e^2}, \qquad \left|\frac{\omega}{k}\right| \ll v_{th}, \tag{7}$$

where λ_e is the electron Debye length and v_{th} is the thermal velocity. A more complete discussion of the dielectric function is given by Nicholson.

Following Nicholson, we ignore the contributions to Eq. (5) from the poles at $\epsilon(k,\omega)=0$, as these contributions die out. The remaining poles are at $\omega=0$ and $\omega=kv$. Again assuming that $f_0(v)$ is Maxwellian, inverse Laplace transforming gives

$$f_1(k,v,t) = \frac{e\sigma}{2\pi\epsilon_0 m}\frac{f_0(v)}{v_{th}^2}\left[\frac{e^{-ikvt}}{k^2\epsilon(k,\omega=kv)} - \frac{1}{k^2\epsilon(k,\omega=0)}\right]. \tag{8}$$

In the limit $v \ll v_{th}$, inverse Fourier transforming gives

$$f_1(x,v,t) = \frac{e\sigma}{2\pi\epsilon_0 m}\frac{f_0(v)}{v_{th}^2} \times \int_{-\infty}^{\infty}dk\left[\frac{e^{ik(x-vt)}}{k^2+k_e^2} - \frac{e^{ikx}}{k^2+k_e^2}\right], \tag{9}$$

where k_e is $1/\lambda_e$, and we have used Eq. (7). Eq. (9) has poles at $k=\pm ik_e$. We close the contour in the upper-half plane, yielding the perturbed distribution function

$$f_1(x,v,t) = \frac{e\sigma}{2\epsilon_0 m}\frac{f_0(v)}{v_{th}^2}\lambda_e\left(e^{-|x-vt|/\lambda_e} - e^{-|x|/\lambda_e}\right) = \frac{e}{\kappa T}f_0(v)\left[\Phi(x) - \Phi(x-vt)\right],$$

$$|v| \ll v_{th} \tag{10}$$

where

$$\Phi(\zeta) = \frac{\sigma\lambda_e}{2\epsilon_0}e^{-|\zeta|/\lambda_e}, \tag{11}$$

T is the plasma temperature, and κ is Boltzmann's constant. Note that $\Phi(\zeta)$ is the standard result for the shielded potential from a charge sheet σ.

For a particle of velocity v, the perturbed distribution function $f_1(x,v,t)$ is time independent only after $\Phi(x-vt)$ goes to zero, i.e. after a time τ such that $|x-v\tau| \gg \lambda_e$. But assuming that the particle starts near $x=0$, this inequality implies that the particle travels a distance substantially greater than a

Debye length λ_e in time τ. As the particles trapped in the potential well can only travel a distance approximately equal to the Debye length before they bounce, the trapped particles make many bounces before their distribution function becomes time independent. Consequently, the unperturbed-orbit technique used to solve the linearized Vlasov equation is violated. There is no time at which the trapped-particle distribution function has reached steady state and the unperturbed-orbit technique is valid.

If trapped particles were not crucial to shielding, all this would be of little importance. That the trapped particles are crucial is easy to see by calculating the trapped-particle density. For the moment, let us ignore the problems with the above derivation, and assume that the potential really is given by Eq. (11). Then the complete distribution function is given by $f(x,v)=\exp[e\Phi(x)/kT]f_0(v)$, where $f_0(v)$ is a Maxwellian. At $x=0$, particles whose velocity is less than $|v|<v_T=\sqrt{2e\Phi(0)/m}$, are trapped, so the density of trapped particles at the origin equals

$$\begin{aligned} n_T &= \int_{-v_T}^{v_T} f_1(x=0,v)\,dv \\ &= n_0 \exp\left[\frac{e\Phi(0)}{kT}\right] \operatorname{erf}\left[\sqrt{\frac{e\Phi(0)}{kT}}\right] \\ &= 2n_0\sqrt{\frac{e\Phi(0)}{\pi kT}} + \mathscr{O}\left[\frac{e\Phi(0)}{kT}\right]^{3/2}, \end{aligned} \tag{12}$$

where erf is the error function and n_0 is the unperturbed density. The density of free particles equals

$$\begin{aligned} n_F &= 2\int_{v_T}^{\infty} f_1(x=0,v)\,dv \\ &= n_0 \exp\left[\frac{e\Phi(0)}{kT}\right] \operatorname{erfc}\left[\sqrt{\frac{e\Phi(0)}{kT}}\right] \\ &= n_0\left\{1 - 2\sqrt{\frac{e\Phi(0)}{\pi kT}} + \frac{e\Phi(0)}{kT} \right. \\ &\quad \left. + \mathscr{O}\left[\frac{e\Phi(0)}{kT}\right]^{3/2}\right\}, \end{aligned} \tag{13}$$

where erfc is the complementary error function. As expected, the total density equals

$$\begin{aligned} n_{tot} = n_T + n_F &= n_0 \exp\left[\frac{e\Phi(0)}{kT}\right] \\ &= n_0\left\{1 + \frac{e\Phi(0)}{kT} + \mathscr{O}\left[\frac{e\Phi(0)}{kT}\right]^{3/2}\right\}, \end{aligned} \tag{14}$$

the unperturbed density times a Boltzmann factor. Obtaining this result is critically dependent on the square root terms in Eqs. (12) and (13) canceling; if these terms do not cancel, n_{tot} will have a $\sqrt{e\Phi/kT}$ term which will *dominate* the desired $e\Phi/kT$ term. In other words, the density of trapped particles is never small as it depends on $\sqrt{e\Phi/kT}$. An error calculating the trapped density has profound consequences for the final answer.

In sum, the problem with the textbook shielding derivation is that by the time the trapped-particle distribution has reached steady state, the solution is no longer valid, and we cannot be sure that we will get the necessary perfect cancellation of the square-root terms. This is not merely an academic exercise; experiments [4] have demonstrated that when the trapped-particle density is anomalous, the sign of the shielding can change; the plasma can *enhance* rather than diminish the fields from test sheet. Some authors [3] even think that this 'anti-shielding' is the default result in one-dimension. Experimentally we find, however, that we usually do get shielding [5,6], implying that the square-root terms usually cancel. Physically, this probably results from the shielded trapped-particle density being close to the original density of particles with $|v|<v_T$. So long as these two densities do not differ more than linearly in Φ, the square root terms will cancel, and the linear term will be as expected. The higher-order terms, however, will not necessarily conspire to give the Boltzmann result, $n_0\exp[e\Phi(0)/kT]$. Experiments have demonstrated that the higher order response depends on how the plasma is prepared and on how the charge is inserted into the plasma [4–6].

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